Hitchin Systems in Supersymmetric Field Theory II

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Last time we claimed that $X_{\mathfrak{g}}[S^1]$ lookws in the limit of energy much less than $\frac{1}{R}$ (R the radius of S^1) we got 5d SYM. But we didn't get to the Hitchin system. This comes from compactifying not on S^1 but on a Riemann surface C. The result $X_{\mathfrak{g}}[C]$ is a 4d field theory which has $\mathrm{ISO}(3,1)$ symmetry and $N = 2$ supersymmetry. This theory is still very complicated, but in the limit where we rescale the metric of C

$$g_C \mapsto tg_C, t \to 0 \tag{1}$$

to zero, we don't remember the metric but only the conformal class of the metric, or equivalently it only depends on C as a Riemann surface.

Q: in what sense do we take a limit of QFTs?

A: we can make sense of the same local operator at different scales t, and then the claim is that correlation functions converge, for example.

Observables in $X_{\mathfrak{g}}$ map to observables in $X_{\mathfrak{g}}[C]$; for example, local operators make to local operators. Surface operators can map to several different things. We can get a surface operator if we take surfaces in $\mathbb{R}^{3,1}$. We can get a line operator if we take the product of a line in $\mathbb{R}^{3,1}$ and a loop in C. And we can get a local operator if we take C itself. 4-manifold operators give yet more operators.

In general, $X_{\mathfrak{g}}[C]$ is not known to be Lagrangian. When it is, that's very interesting, but for the most part we'll ignore this.

The Hitchin system arises after making one more reduction. Let's consider the compactification $X_{\mathfrak{g}}[C \times S^1]$. After making some clever choices we'll get a 3d $N = 4$ theory. There are two regimes to study this theory on: in one, S^1 is much smaller than C, or symbolically

$$|S^1) \ll |C| \tag{2}$$

and in the other,

$$|S^1| \gg |C|. \tag{3}$$

We'll get some insights from each limit, but it turns out that we can use these insights simultaneously in the general case.

Q: didn't we take $|C| = 0$?

A: we could take that limit if we wanted to, but by default we're choosing not to. Let's write $X_{\mathfrak{g}}[C, 0]$ for the size-zero limit. Eventually we'll take the limit in which both S^1 and C are small.

Let's start with the small-S^1 regime. We can think of the resulting theory as starting from $X_{\mathfrak{g}}[S^1]$, taking the small-S^1 (low-energy) limit, and compactifying the resulting theory (5d SYM) on C. Recall that the fields of 5d SYM were

1. A connection $D = d + A$,

2. $\mathfrak{g}$-valued functions $\Phi^i, 1 \leq i \leq 5$ which together live in the defining representation of $SO(5)$.

After compactifying on C, the $SO(5)$ symmetry is broken. Φ^4 and Φ^5 together are now repackaged as a complex 1-form

$$\varphi = (\Phi^4 + i\Phi^5)\, dz \tag{4}$$

on C.

What is the low-energy (energies much less than $\frac{1}{|C|}$) physics of this theory? In general, this question can't be answered by purely classical considerations, but 5d SYM is known to be infrared free (so the moduli space of classical vacua needs no quantum corrections; this is specific to 5 dimensions), which allows us to use classical field theory to answer this question.

For starters, we can ask for the classical field configurations / vacua invariant under the remaining ISO(2, 1) symmetry and under supersymmetry. For simplicity, we'll assume that $\Phi^1 = \Phi^2 = \Phi^3 = 0$. The equations of motion are that the connection D and the 1-form φ satisfy

$$F_D + R^2[\phi, \phi^\dagger] = 0 \tag{5}$$

and

$$\overline{\partial}_D \varphi = 0. \tag{6}$$

These are Hitchin's equations, and the moduli space of classical vacua $M_G(C)$ (solutions to the above equations modulo gauge transformations) is the Hitchin integrable system.

If we compute the correlation functions of $X_{\mathfrak{g}}[C \times S^1]$ on $\mathbb{R}^{2,1}$, the result are not numbers, but functions on the moduli space of vacua. One way to see this is that the path integrals defining the correlation functions require some boundary conditions. We can take the boundary condition to be that the fields (D, φ) approach some fixed values $(D_0, \varphi_0) \in U$ as we go to infinity in $\mathbb{R}^{2,1}$.

Q: what happened to Φ^1, Φ^2, Φ^3?

A: supersymmetry also imposes equations on these, which are that

$$D\Phi = 0, [\varphi, \Phi^i] = 0, [\Phi^i, \Phi^j] = 0. \tag{7}$$

For generic D, φ, these equations have no nonzero solutions, so it wasn't so important that we ignored them. In general, $M_G(C)$ (the Coulomb branch) is one branch of the moduli space of classical vacua, and at special points there may be other branches (mixed branches or Higgs branches).

Q: what are branches? For an algebraic geometer, are they irreducible components?

A: at least loosely. One way to distinguish branches is to look at how R-symmetry acts on them.

Q: how many branches are there? Are there theories where we know the number?

A: there may be many (finitely many) branches. There are some theories without Higgs branches. In general there's not a lot that we can say.

Q: why is the Hitchin moduli space of Higgs bundles the Coulomb branch and not a Higgs branch?

A: good question! I don't know.

The low-energy limit of $X_{\mathfrak{g}}[C \times S^1]$ (where we take everything to be small, with S^1 small first) is the sigma model with target the moduli space of classical vacua, but for now we'll restrict our attention to the Coulomb branch / Hitchin moduli space $M_G(C)$. This means that the fields are maps

$$\sigma : \mathbb{R}^{2,1} \to M_G(C) \tag{8}$$

and there are also some fermions around. The $N = 4$ SUSY requires that $M_C(G)$ have a hyperkähler metric, which is true of $M_G(C)$.

More precisely, when we talk about low-energy limits we are working with respect to a particular vacuum. If we look at a point in the Coulomb branch, then at low enough energies we'll get a sigma model that only involves the Coulomb branch.

We'd like to get some new information about $M_G(C)$ from this point of view. We'll do this by taking limits in the other order: first we'll think about the compactification $X_{\mathfrak{g}}[C]$, then we'll take the small-C (low-energy) limit, then we'll compactify on S^1, then we'll take the small-S^1 (low-energy) limit.

So first we'll have to understand the low energy limit of $X_{\mathfrak{g}}[C]$. Like the 3d theory above, this 4d theory has a moduli space of vacua, and correlation functions etc. are not numbers but functions on this moduli space.

Q: is asking for the moduli space of vacua a mathematically well-defined question? What's wrong with the definition Ben-Zvi gave as Spec of local operators?

A: there's a question of whether we take all local operators or SUSY local operators. Taking all local operators might work. Costello: Witten suggested a definition in terms of functions on local operators which are only asymptotically homomorphisms as the local operators get farther apart.

The Coulomb branch of the moduli space of vacua of $X_{\mathfrak{g}}[C]$ is the base B of Hitchin's integrable system. For example, if $\mathfrak{g}$ is of type A_{k-1}, then

$$B = \bigoplus_{r=2}^{k} H^0(C, K_C^{\otimes r}). \tag{9}$$

Some physics words: the Coulomb branch has a singular locus. At any nonsingular point in the Coulomb branch, there is an electromagnetic charge lattice, and there are some distinguished central charges.

Some math words: the Coulomb branch is a complex manifold B. It has a distinguished divisor B_{sing}. There is a local system of lattices

$$\Gamma \to B_{reg} = B \setminus B_{sing}. \tag{10}$$

And there are fiberwise homomorphisms $Z : \Gamma \to \mathbb{C}$ varying holomorphically over B_{reg}.

In the case of the Hitchin integrable system, at every point $u \in B$ there is a spectral curve Σ_u. B_{sing} is the discriminant locus at which the spectral curve becomes singular. The local system of lattices is the first homology $H_1(\Sigma_u, \mathbb{Z})$ of the spectral curves, and the fiberwise homomorphisms are

$$Z_\gamma = \oint_\gamma \lambda. \tag{11}$$

It turns out that the low-energy physics of this theory only depends on this local system of lattices and on these fiberwise homomorphisms. The key statement, due to Seiberg and Witten, is that for any 4d $N = 2$ SUSY gauge theory, if $u \in B_{reg}$, then at energy scales lower than

$$\min_{\gamma \in \Gamma} |Z_\gamma| \tag{12}$$

the low-energy physics is given by 4d SYM with abelian gauge group $G = U(1)^r$ where $r = \dim B$. Next we'd like to compactify this theory on S^1 and compare it to our sigma model into the Hitchin moduli space $M_G(C)$ to learn many things about the Hitchin moduli space.

Q: are these singular points the points at which the Higgs branches, if any, come out?

A: yes.

Q: what are the constraints on the central charges?

A: the charge lattice comes with an antisymmetric pairing $\langle -, - \rangle$; in this case this pairing comes from the intersection pairing on $H_1(\Sigma_u, \mathbb{Z})$. The extra condition is that

$$\langle dZ, dZ \rangle = 0. \tag{13}$$

Q: how do you justify the claim that the small-S^1 and small-C limits give the same answer? Similarly, how do you justify the claim that in the small-C limit the answer only depends on the conformal class of C?

A: arguments like the following. Take $\mathfrak{g}$ to be of type A_N. As $N \to \infty$ this theory is supposed to have a dual description involving gravity coming from the holographic principle. In that dual description, there is a direction related to energy scales. Anderson-Bobev-Beem-Rastelli started with an arbitrary metric on C, solved the gravity equations, and looked at how the metric changed. The metric appeared to flow to the hyperbolic metric.